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LUNAR LANDING AND TAKE-OFF

Technical Report 63-2

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Abstract

Assumption of the uniform gravity field was justified by computing errors due to this assumption and establishing the limits within which the simplified computations should be performed. Vertical take-off and ascent was analyzed for different values of a constant propellant flow. The analysis of the vertical descent and landing was extended to the cases of a constant propellant flow and thrust, to the modulated and intermittent thrust. Typical numerical examples were computed without embarking on a generalized optimization problem, but indicating the ways of minimizing the propellant consumption.

CHAPTER I

Physical Conditions

1.1 The Moon is not an exactly spherical body. It has an equatorial bulge pointing towards the Earth, hence its shape can be approximated as an tri-axial ellipsoid.

In the present report, however, the Moon is considered as a spherical body of a radius R. There is no sea-level on the Moon and lunar ground level gravity acceleration corresponds to the level of large maria, like Mare Imbrium. The slow axial rotation of the Moon allows to neglect the centrifugal effect on the gravity. The following data have been accepted for computations:

Earth's true equatorial gravity	$g_e' = 32.2284$ ft/sec^2
Earth's standard gravity measured at $\lambda = 46^0$	$g_e = 32.174$ ft/sec^2
The mass of the Moon	$M_m = 0.0123\ M_e$
Lunar ground gravity acceleration	$g_o = 0.164\ g_e'$
Lunar ground gravity acceleration	$g_o = 5.3$ ft/sec^2

Lunar mean radius R = 1080 stat. miles = 1740 km.

Sidereal period of axial rotation T = 27.32166 days

Angular velocity of rotation $\omega = \frac{2\pi}{T} = 0.23 \frac{rad}{day}\ 2.66 \times 10^{-6} \frac{rad}{sec}$

Tow velocity at the equator $\omega R = 2.88 \times 10^{-3} \frac{miles}{sec.} = 15.2 \frac{feet}{sec.}$

Gravity constant $g_o R^2 \cong kM = 1.17 \times 10^3 \frac{(\text{miles})^3}{\text{sec}^2}$

Square of ground circular velocity $V_c{}^2 = g_o R = 1.084 \frac{\text{mi}^2}{\text{sec}^2}$

Lunar ground circular velocity $V_c = 1.041$ mi/sec = 5500 ft/sec.

Lunar ground escape velocity $V_{es} = 1.472$ mi/sec = 7770 ft/sec.

CHAPTER II

Errors Due To Uniform Field Assumption

2.1 Assumption of a uniform gravity field for the final phase of the lunar landing or for the initial phase of the take-off, can be justified if we establish the volume within which the error due to this assumption is negligible. If we extend the simplifying assumptions beyond this volume a method should be provided to find easily the corrections for computed values of range, velocity etc.

Let us examine first the variation of the lunar gravity acceleration in the vertical direction. As there is no sea-level value for this acceleration, what we call a ground value g_o may not have the same value on the lunar surface. In the present paper these local divergencies are neglected and the ground level of g is assumed to be $g_o = 5.3$ ft/sec^2. Assuming sphericity of Ahe Moon and concentric density distribution we express the gravity variation with the altitude y over a hypothetical ground level.

$$g = g_o \left(\frac{R}{R + y}\right)^2 = g_o \left(1 + \frac{y}{R}\right)^{-2} =$$

$$= g_o \left[1 - 2\frac{y}{R} + \frac{2.3}{1.2}\left(\frac{y}{R}\right)^2 - \frac{2.3.4}{1.2.3}\left(\frac{y}{R}\right)^3 + \frac{2.3.4.5}{1.2.3.4}\left(\frac{y}{R}\right)^4 - \ldots\right.$$

Assume as a mean lunar radius R = 1080 stat. miles and find the maximum relative error $\frac{\Delta g}{g_o}$ in replacing the variable g by a constant value g_o up to the altitude y = 10.8 st. miles = 57,000 ft.

$$\frac{\Delta g}{g_o} = \frac{g_o - g}{g_o} = 2\,\frac{y}{R}\left[1 - \frac{3}{2}\left(\frac{y}{R}\right) + 2\left(\frac{y}{R}\right)^2 - \frac{5}{2}\left(\frac{y}{R}\right)^3 - \right]$$

if we take $\frac{y}{R} = \frac{1}{100}$,

$$\frac{\Delta g}{g_o} = \frac{2}{100}\left(1 - \frac{3}{2 \times 10^2} + \frac{2}{10^4} - \frac{5}{2 \times 10^6} + \,..\right)$$

$$= \frac{2}{100}\left(1 - 0.015 + 0.002 - 0.0000025 + \ldots\right) = \frac{2}{100} \times 0.9852 = 0.0197$$

hence the error is slightly less than 2 percent of the initial value. By accepting this error we should investigate the time and altitude errors due to this approximation.

2.2 Now let us consider the errors in horizontal motion (range error).

In the uniform gravity field we assume also that a planet has an infinite radius, and hence its surface is an infinite plane passing through the point selected as an origin. The sphericity of the planet produces the inclination of local vertical with respect to OY at the origin. The range is measured along the plane tangent to the surface at O, and the curvature of the surface may increase it as shown on Figure 1. The point P where a landing may theoretically occur is actually at an altitude Δy above the ground level. For any arbitrary horizontal distance x let us compute the angular ($\Delta\gamma$) and altitude (Δy) errors.

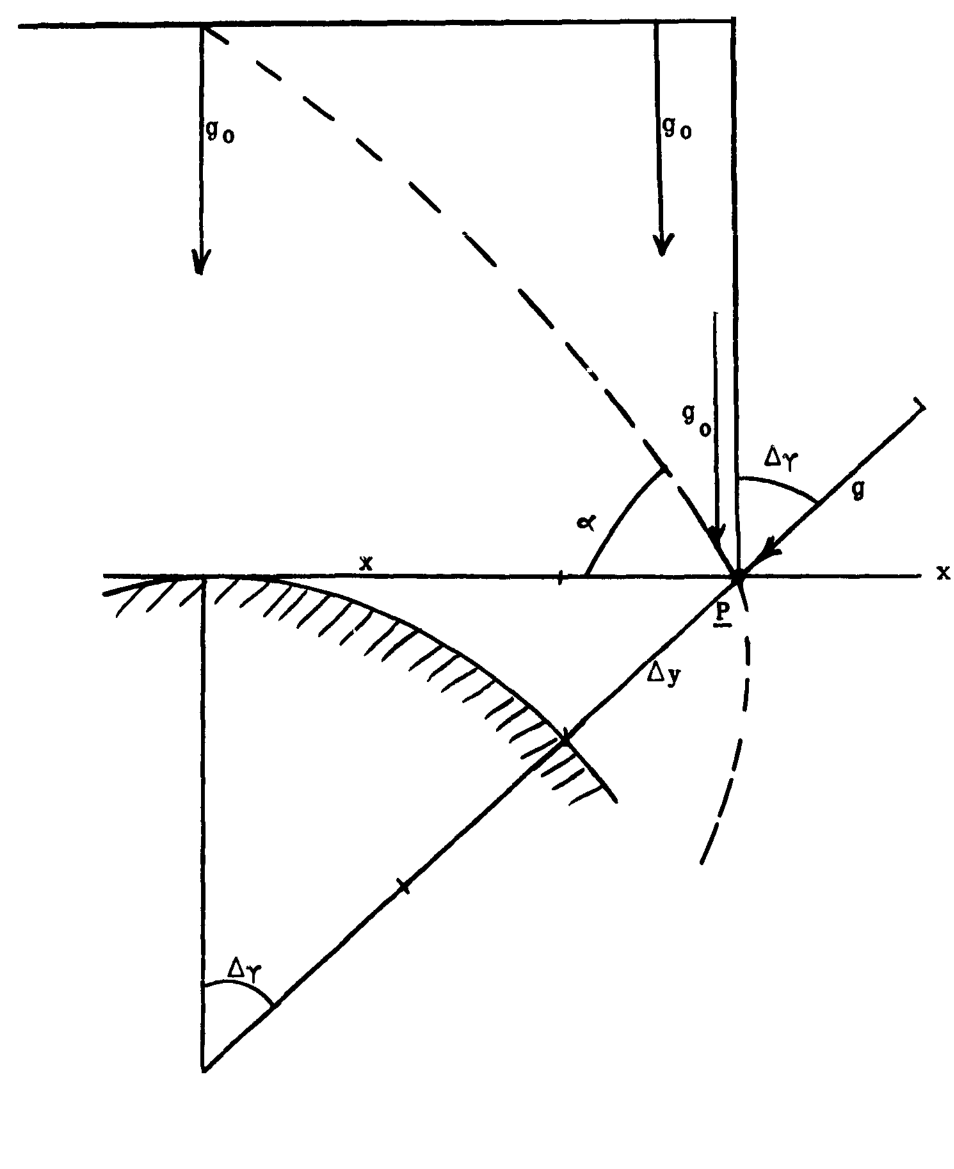


Figure 1

$$\tan \Delta\gamma = \frac{x}{R}$$

$$(R + \Delta y)^2 = R^2 + x^2$$

or $$2\,\frac{\Delta y}{R} + \frac{\Delta y^2}{R^2} = \frac{x^2}{R^2}$$

neglecting the higher powers of $\left(\frac{\Delta y}{R}\right)$ we finally have

$$\frac{\Delta y}{R} = \frac{x^2}{2R^2}$$

TABLE I

x	2	4	6	8	10	12	14	Stat.Mi.
$\frac{x}{R} \times 10^3$	1.85	3.70	5.56	7.41	9.26	11.11	12.97	
$\Delta\gamma$	0°06'	0°13'	0°19'	0°25'	0°32'	0°38'	0°45'	
$\frac{\Delta y}{R} \times 10^6$	1.71	6.85	15.45	27.45	42.87	61.72	84.11	
$\Delta y \times 10^3$	1.848	7.4	16.69	29.62	46.28	66.62	90.90	St.Mi.
Δy	9.8	39.1	88.1	156.5	244	352	480	Ft.

From the Figure 2 an approximate extension of the range due to sphericity can be estimated as $\Delta x = \Delta y \cot \alpha$, where $\alpha = 180^0 - \alpha_F$, α_F is the fall angle.

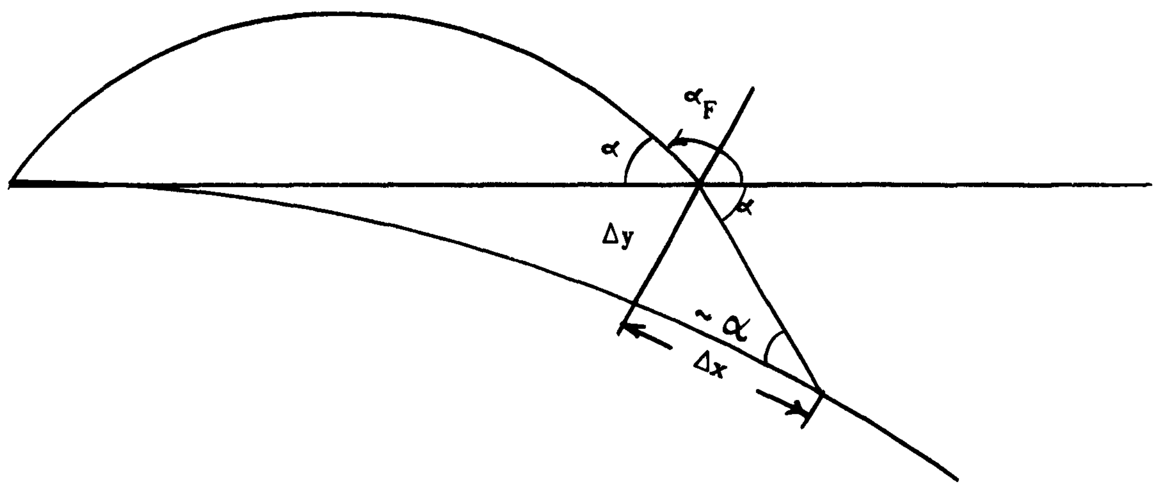


Figure 2

2.3 The errors in the estimation of time and velocity are computed only for the case of a free vertical fall.

In a uniform gravity field

$$\ddot{y} = - g_0 \quad ;$$

or

$$\frac{d^2y}{dt^2} = \dot{y}\,\frac{d\dot{y}}{dy} = - g_0 \ .$$

The integration yields

$$\frac{\dot{y}^2 - \dot{y}_0{}^2}{2} = - g_0 \,(y - y_0) \ .$$

If the initial speed $\dot{y}_0 = 0$, and if $y = 0$ that is the fall ends at ground level we obtain the impact velocity

$$\dot{y}_i{}^2 = 2\, g_0 y_0 \ .$$

Consider now a free fall in a variable gravity field, obeying inverse square law.

$$\dot{y}\,\frac{dy}{dy} = - g_0 \,\frac{R^2}{(R + y)^2} \quad ,$$

hence

$$\frac{\dot{y}^2 - \dot{y}_0{}^2}{2} = g_0 R^2 \left[\frac{1}{R + y} - \frac{1}{R + y_0} \right] = g_0 R^2 \,\frac{y_0 - y}{(R + y)(R + y_0)}$$

If $y = 0$ and $\dot{y}_0 = 0$, we obtain

$$\dot{y}^2 = 2\, g_0 R \,\frac{y_0}{R + y_0}$$

The difference between the vertical impact speed $\dot{y}_v$ of variable field and that of the uniform one will be found from the equation

$$\dot{y}^2 - \dot{y}_v^{\,2} = (\dot{y} - \dot{y}_v)(\dot{y} + y_v) = 2\, g_o y_o \left[1 - \frac{R}{R + y_o} \right] = \frac{2\, g_o y_o^{\,2}}{R + y_o}$$

Hence the error due to simplified assumption is of the order

$$\Delta\dot{y} = \dot{y} - \dot{y}_v = \frac{2\, g_o y_o^{\,2}}{\left(R + y_o\right) \sqrt{2\, g_o y_o} \left[1 + \sqrt{\frac{R}{R + y_o}} \right]} =$$

$$= \frac{y_o}{\left(R + y_o\right)} \; \frac{\sqrt{2\, g_o y_o}}{\left(1 + \sqrt{\frac{R}{R + y}}\right)} = \frac{\frac{y_o}{R}}{\left(1 + \frac{y_o}{R}\right)} \cdot \frac{\sqrt{2\, g_o R} \sqrt{\frac{y_o}{R}}}{\left(1 + \sqrt{\frac{1}{1 + \frac{y_o}{R}}}\right)}$$

$$= \sqrt{2\, g_o R} \; \frac{\left(\frac{y_o}{R}\right)^{3/2}}{\left(\sqrt{1 + \frac{y_o}{R}}\right)\left(1 + \sqrt{1 + \frac{y_o}{R}}\right)} \; . \; .$$

By using serial expansion the error function can be given the following form

$$\Delta\dot{y} = \sqrt{2 g_o R} \left(\frac{y_o}{R}\right)^{3/2} \left(1 + \frac{y_o}{R}\right)^{-1/2} \left[1 + \left(1 + \frac{y_o}{R}\right)^{1/2} \right]^{-1} =$$

$$\cong \frac{1}{2} \sqrt{2\, g_o R} \left(\frac{y_o}{R}\right)^{3/2} \left(1 - \frac{1}{2} \frac{y_o}{R}\right) \left(1 - \frac{1}{4} \frac{y_o}{R}\right)$$

$$\cong \frac{1}{2} \sqrt{2\, g_o R} \left(\frac{y_o}{R}\right)^{3/2}$$

For the Moon $\sqrt{2\, g_o R}$ = escape velocity from the ground level = 7770 ft/sec.

The following table of impact velocity errors has been prepared assuming a free fall from the altitude y_o with the initial zero velocity. The correct values of impact vertical velocity have also been computed from the formula

$$\dot{y}_v = \sqrt{2\, g_o R}\ \sqrt{\frac{y_o/R}{1 + y_o/R}}$$

and the relative error estimated ...

$$\frac{\Delta \dot{y}}{\dot{y}_v} = \left(\frac{1}{2}\right)\left(\frac{y_o}{R}\right)\sqrt{1 + y_o/R}$$

TABLE II

y_o	2.7	5.4	8.1	10.8	16.2	21.6	St.Mi.
y_o	14280	25560	42800	57100	85500	114000	Ft.
$\frac{y_o}{R}$	.0025	.0050	.0075	.0100	.015	.02	
$\sqrt{1 + \frac{y_o}{R}}$	1.00125	1.00250	1.00374	1.00499	1.00747	1.00995	
$\frac{\Delta \dot{y}}{\dot{y}_v}$	1.25	2.51	3.77	5.03	7.56	10.1	x 10^{-3}
$\dot{y}_v$	386	548	670	774	945	1090	Ft/sec

So the velocity error is usually less than one percent for $y_0 \leq 100{,}000$ ft. If slide rule is used the difference $\Delta\dot{y}$ should be estimated from the formula and not from the difference $\dot{y} - \dot{y}_v$. In the latter case the subtraction of two almost equal numbers greatly magnifies the error.

2.4 The time of free fall from the altitude y_0 in the uniform gravity field is estimated from the formula

$$t = \sqrt{\frac{2\,y_0}{g}} = \frac{1}{\sqrt{2.65}}\sqrt{y_0} = \frac{1}{1.628}\sqrt{y_0} = 0.614\sqrt{y_0}$$

in seconds if the initial zero vertical velocity is assumed, and if y_0 is given in feet.

In the variable gravity field the time t_v is estimated from the Kepler's equation applied to a rectilinear ellipse, that is with an eccentricity e approaching unity: $e \rightarrow 1.0$. An assumption is made that in the vicinity of ground level the velocity is less than its escape value, then t and E are measured from the apofocus of an elliptic orbit

$$n\left(t_v - t_0\right) = E + e \sin E,$$

where

$$n^2 = \frac{g_0 R^2}{a^3} \quad ; \text{ and}$$

the semi-major axis a can be defined from the initial conditions

$$a = \frac{g_0 R^2}{2\,g_0 R\,\dfrac{R}{R + y_0} - V_0^{\,2}} \quad ;$$

and as $V_0 = 0$, we obtain $a = \frac{R + y_0}{2}$, that is the perifocus lies at the Moon's center.

$$n^2 = \frac{8\ g_0 R^2}{\left(R + y_0\right)^3} \ ; \quad n = \frac{2\ R}{\left(R + y_0\right)} \sqrt{\frac{2\ g_0}{R + y_0}} =$$

$$= \frac{2\sqrt{2\ g_0}}{\sqrt{R}\left(1 + \frac{y_0}{R}\right)^{3/2}} .$$

Substitution of numerical values yields

$$n = \frac{2\sqrt{10.60}}{\sqrt{\left(1080\right) \times \left(5280\right)}} \left(1 + \frac{y_0}{R}\right)^{-3/2} =$$

$$= 2.727 \times 10^{-3} \times \left(1 + \frac{y_0}{R}\right)^{-3/2} \frac{1}{\text{sec}} .$$

For $e = 1$ the ellipse is reduced to a straight line, the eccentric anomaly E is computed from the Figure 4.

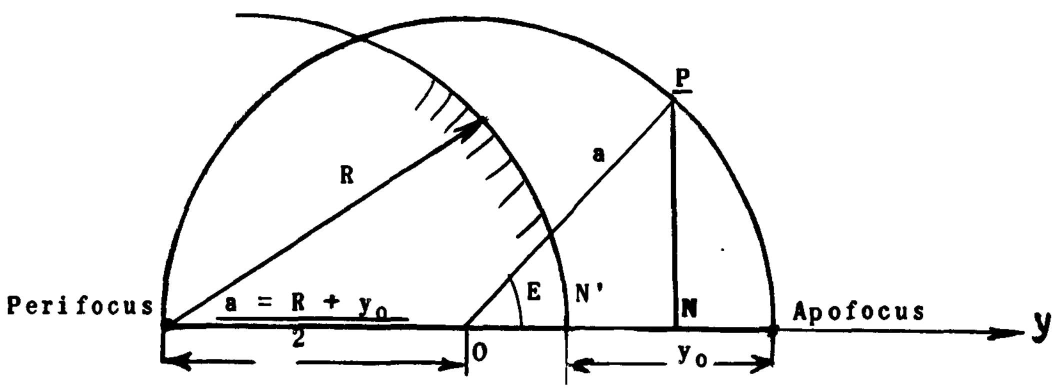


Figure 4

At the ground level

$$ON' = a - y_o = \frac{R + y_o}{2} - y_o = \frac{R - y_o}{2} \quad ;$$

$$\cos E = \frac{ON}{a} = \frac{R - y_o}{R + y_o} \quad ;$$

$$\sin E = \sqrt{1 - \cos^2 E} = \sqrt{1 - \left(\frac{R - y_o}{R + y_o}\right)^2} = \frac{2\sqrt{y_o R}}{R + y_o}$$

$$= \frac{2\sqrt{y_o/R}}{1 + y_o/R} \quad ;$$

$$E = \text{arc sin } \frac{2\sqrt{y_o/R}}{1 + y_o/R.} \quad ;$$

Substitute into the Kepler's equation, taking $t_o = 0$, and $e = 1$,

$$t_v = \frac{1}{n}\left(E + e \sin E\right) = 3.667 \times 10^2 \times \left(1 + \frac{y_o}{R}\right)^{3/2}\left(E + \sin E\right)$$

The error due to the uniform field assumption is given by the difference.

$$\Delta t = t_v - t = - 0.614 \sqrt{y_o}\ 3.667 \times 10^2 \times \left(1 + \frac{y_o}{R}\right)^{3/2}\left(E + \sin E\right)$$

The results are given in the Table III.

TABLE III

y_o	14280	28560	42800	57100	114000	Ft	(1)
$\frac{y_o}{R}$	0.0025	0.0050	.0075	0.010	0.02		(2)
$\sqrt{\frac{y_o}{R}}$	5	7.07	8.66	10	14.14	x 10^{-2}	(3)
$1 + \frac{y_o}{R}$	1.0025	1.0050	1.0075	1.010	1.02		(4)
$\left(1 + \frac{y_o}{R}\right)^{3/2}$	1.0037	1.0075	1.0113	1.0150	1.0300		(5)
sin E	.09975	.1408	.1720	.1980	.2774		(6)
E	.09995	.1412	.1728	.1993	.2810	rad.	(7)
E + sin E	.19970	.2820	.3448	.3973	.5584		(8)
(5) x (8)	.2005	.2840	.3493	.4038	.5750		(9)
t_v	73.5	104.2	128.0	148.0	211	Sec.	(10)
$\sqrt{y_o}$	119.8	169.0	206.9	239	337.6		(11)
t	73.5	103.8	127	146.8	207	Sec.	(12)
Δt	0	0.4	1.0	1.2	4	Sec.	(13)
$\Delta t/t$	0	3.85	7.8	8.1	19	x 10^{-3}	(14)

2.5 Having established the limits of errors due to the assumption of a uniform gravity field we may consider a cylindrical box whose base is resting on the lunar surface. The initial point of any trajectory lies on the vertical axis of symmetry of the box; whose radius is 14 stat. miles $\cong$ 74000 ft., its height = 22 stat. miles = 114000 ft. If the initial and end point of a trajectory lie within this box the errors due to simplifying assumptions are limited by the following values:

$\Delta\gamma \leq 0^{\circ}45'$

$\Delta y \leq 480$ feet (altitude of the impactfor horizontal range)

$$\frac{\Delta\dot{y}}{\dot{y}} \leq 1 \times 10^{-2}$$

$$\frac{\Delta t}{t} \leq 2 \times 10^{-2}$$

If a further sacrifice of accuracy is permitted by extending the dimensions of the box, the following formulas enable us to estimate the order of magnitude of the error.

$$\frac{\Delta y}{R} = \frac{x^2}{2\ R^2} \quad ;$$

$$\Delta x = (\Delta y) \cot (180^{\circ} - \alpha_F)$$

$$\Delta\dot{y} = \frac{1}{2}\sqrt{2\ g_o R}\left(\frac{y_o}{R}\right)^{3/2} = 3885\left(\frac{y_o}{R}\right)^{3/2} \text{ ft/sec.}$$

$$\frac{\Delta\dot{y}}{\dot{y}} = \frac{1}{2}\left(\frac{y_o}{R}\right)\sqrt{1 + \frac{y_o}{R}}$$

$$\Delta t = 3.667 \times 10^2 \times \left(1 + \frac{y_o}{R}\right)^{3/2}\left(E + \sin E\right) - 0.614\sqrt{y_o}$$

where

$$\sin E = \frac{2}{1 + \frac{y_o}{R}} \sqrt{\frac{y_o}{R}} \quad , \; y_o \text{ in feet and } \Delta t \text{ in seconds}$$

$$\frac{\Delta g}{g_o} = 2\left(\frac{y}{R}\right)\left[1 - \frac{3}{2}\left(\frac{y}{R}\right) + 2\left(\frac{y}{R}\right)^2 - \frac{5}{2}\left(\frac{y}{R}\right)^3 + \ldots\right]$$

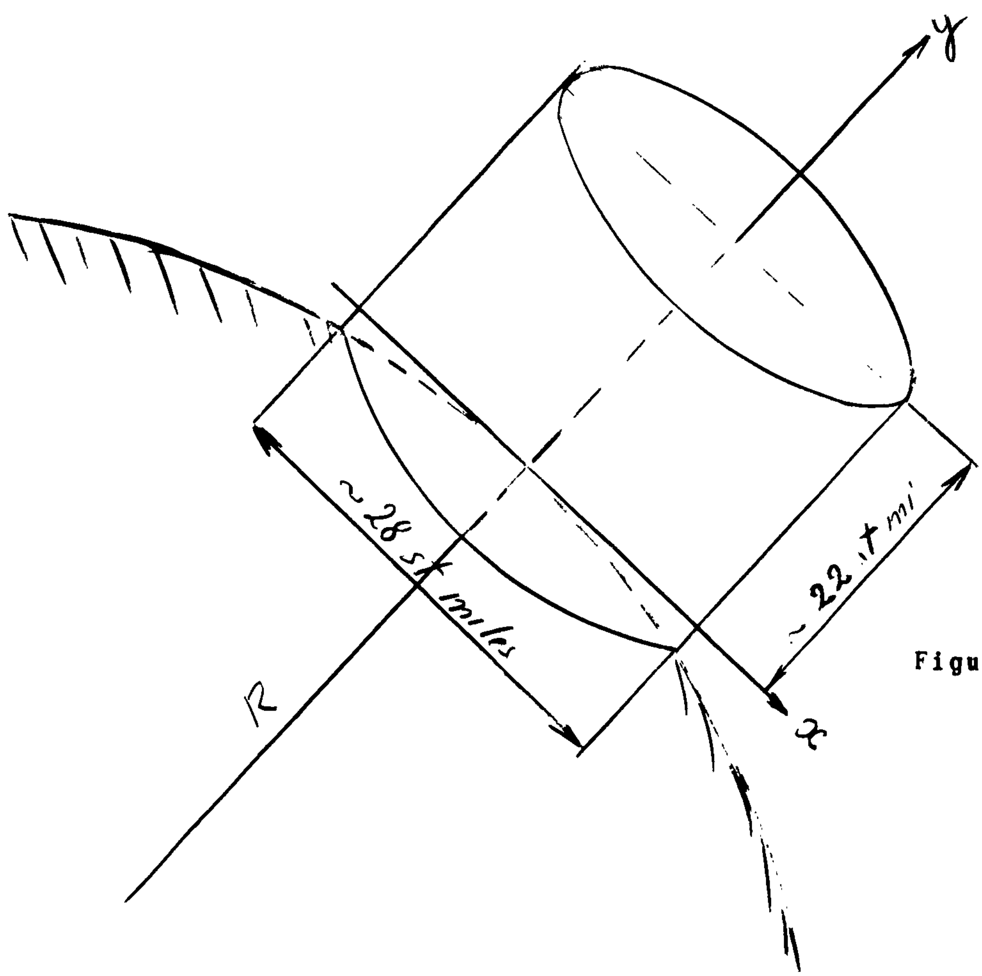


Figure 5

CHAPTER III

Powered Ascent

3.1 The total thrust of a rocket engine is given by the expression

$$F = \dot{m}\, u + A_e \left(p_e - p_o \right) \tag{3-1}$$

where $\dot{m}$ is the mass flow of the propellant in slugs per sec

u the actual exhaust velocity in ft/sec measured relative to the rocket at the exit

A_e = the exit area in ft^2

p_e = the exit pressure in lb/ft^2.

p_o = surrounding pressure

In the vacuum $p_o = 0$ and the thrust assumes a value

$$F = \dot{m}\, u + A_e\, p_e \tag{3-2}$$

which can be replaced by

$$F = \dot{m}\, u_{eff}.$$

where u_{eff} is the effective exhaust velocity found from the equation

$$u_{eff} = \dot{u} + \frac{A_e\, p_e}{\dot{m}} \tag{3-3}$$

In further development the index eff. will be dropped and u will always mean an effective velocity. Hence $F = \dot{m}\, u$.

The expression for the specific thrust I of a rocket is obtained from the equation:

$$F = \dot{m}\ u = I\ \dot{w} = \dot{m}\ g_e\ I \quad ; \qquad (3\text{-}4)$$

where:

F -- the total thrust of the rocket engine in pounds.

u -- the effective exhaust velocity at the nozzle's exit in ft/sec. The term $\dot{m}$ u includes the exit pressure term.

$\dot{m}$ -- the mass flow of the propellant = fuel + oxydizer in slugs/sec.

I -- specific thrust in $\dfrac{\text{lb (Thrust)}}{\text{lb/sec (propellant)}}$ = sec.

$\dot{w}$ -- the propellant consumption in lb/sec.

hence

$$u = I\ \frac{\dot{w}}{\dot{m}} = I\ \frac{\dot{m}\ g_e}{\dot{m}} = I\ g_e \qquad (3\text{-}5)$$

g_e -- is a constant coefficient giving the ratio between the specific thrust and exhaust velocity ($I = \dfrac{u}{g_e}$), it is not affected by varying gravity field.

g_e is not the true gravity acceleration but is the standard NACA gravity acceleration corresponding to that measured on the surface of the Earth at approximately $\lambda = 46^{o}$N, hence it corresponds to the conditions at which rocket fuel weight is measured on the Earth. Its value = g_e = 32.174 ft/sec^2.

3.2 Consider now the vertical ascent of a rocket in airless conditions. The forces acting at every instant on the remaining mass M of the rocket are those of the thrust and weight, hence

$$M \frac{dV}{dt} = F - Mg$$

In this problem a constant $g = g_0$ is assumed, provided the altitude is below an imposed value, the exhaust effective speed u is also assumed to be constant, hence:

$$M \frac{dV}{dt} = \dot{m}\, u - Mg_0,$$

or

$$M \frac{dV}{dt} = - u \frac{dM}{dt} - Mg_0, \quad \text{as } \dot{m} = - \frac{dM}{dt} \quad ,$$

hence

$$dV = - u \frac{dM}{M} - g_0 \, dt \quad ;$$

The integration yields:

$$V - V_0 = u \ln \frac{M_0}{M} - g_0 \; t = I \; g_e \ln \frac{M_0}{M} - g_0 \; t \qquad (3\text{-}6)$$

If $V_0 = 0$, the equation assumes the form:

$$V = u \ln \frac{M_0}{M} - g_0 \; t. \qquad (3\text{-}7)$$

where $\ln$ is the natural $\log_e = 2.3026 \log_{10}$, g_0 has been defined as the local ground gravity, M_0 is the initial mass of the rocket at time $t = 0$, M is the terminal mass of the rocket at time $t = t$. The ratio of masses can be replaced by the ratio

of weights whatever the gravity field is.

$$\frac{M_o}{M} = \frac{M_o\ g}{Mg} = \frac{W_o}{W} \quad . \tag{3-8}$$

Insofar no assumptions have been made as to the rate of the propellant consumption, and so the equation (3-6) and (3-7) can be applied to the cases of constant or variable fuel flow. Only a constant u and consequently a constant I has been postulated.

If the rate of the fuel flow is constant and equal to a fixed fraction of the initial mass, we write

$\dot{m} = K\ M_o$, and the mass M after t seconds since the ignition becomes

$$M = M_o\ (1 - Kt), \text{ conversely } W = W_o\ (1 - Kt)$$

3.3 Assume now that the rocket ejects a constant mass $\dot{m} = K\ M_o$ per second, where M_o is the initial mass of the rocket. Let M_1 be the mass of the empty rocket, hence $M_o = M_1 + \text{fuel}$. Show which conditions must be fulfilled:

a) that the rocket can rise at once after the ignition.

b) that it can rise at all.

If it can rise vertically at once find its maximum velocity and maximum height, assuming a constant gravity acceleration g_o.

A rocket can rise at once if its thrust is greater than the initial weight

$$F > W_o$$

or

$$K\ M_o\ u > M_o\ g_o$$

$$K\,u > g_o \text{ or } I\,g_e > \frac{g_o}{K}\,,\ I\,g_e - \frac{g_o}{K} > 0$$

It can rise after some time, if its thrust is at least greater than its empty weight

$$K\,M_o\,u > M_1\,g_o$$

$$K\,u > \frac{M_1}{M_o}\,g_o\,.$$

Assuming $K\,u > g_o$, we can compute the ascension from the moment of ignition. From the previous sections, the vertical speed is at any moment

$$V = V_o - g_o\,t + u\,\ln\frac{M_o}{M}\quad ;$$

If $\quad V_o = 0 \text{ and } M = M_o - K\,M_o\,t = M_o\,(1 - Kt)\,,$

then

$$V = -\,g_o\,t - u\,\ln\,(1 - Kt) = -\,g_o\,t - I\,g_e\,\ln\,(1-Kt) \tag{3-9}$$

The acceleration at any time t of powered flight is

$$\frac{dV}{dt} = -\,g_o + \frac{K\,u}{1 - Kt}\,,$$ and the second derivative of V $$\frac{d^2V}{dt^2} = +\,\frac{K^2\,u}{(1 - Kt)^2}$$

The mathematical extremum of velocity is attained when $\frac{dV}{dt} = 0$, the corresponding value of t is

$$1 = Kt_m = \frac{Ku}{g_o}\quad ;$$

$t_m = \frac{1}{K} - \frac{u}{g_o}$, this value corresponds, however,

to a minimum value of V, as the second derivative of V shows: this is apparently contradictory to the fact that $\frac{dV}{dt}$ is positive and increasing with t.

The paradox is explained by taking notice that t_m is negative, because $\frac{1}{K} - \frac{u}{g_o} = \frac{g_o - K u}{K g_o} < 0$. So this minimum V lies in the past time before the ignition (and motion) started.

During the ascension, insofar $K u > g_o$ the velocity and acceleration (in the vacuum) are both increasing until the exhaustion of fuel, hence the maximum value of the velocity will be attained at a certain time t_1, when $M_o (1 - Kt_1) = M_1$, therefore

$$1 - Kt_1 = \frac{M_1}{M_o} .$$

and

$$t_1 = \frac{1}{K} (1 - \frac{M_1}{M_o}) = \frac{1}{K} (1 - \frac{W_1}{W_o}) .$$

Hence

$$V_{max} = - \frac{g_o}{K} (1 - \frac{M_1}{M_o}) + u \, \ell n \frac{M_o}{M_1} =$$

$$= - \frac{g_o}{K} (1 - \frac{W_1}{W_o}) + I \, g_e \, \ell n \frac{W_o}{W_1} . \qquad (3\text{-}10)$$

The value of the acceleration at this moment also attains its maximum value:

$$(\frac{dV}{dt})_{t_1} = - g_o + \frac{K M_o u}{M_1} =$$

$$= - g_o + \frac{K W_o}{W_1} I \, g_e \qquad (3\text{-}11)$$

Hence

$$\frac{(dV/dt)_1}{g_e} = -0.164 + K\ I\ (\frac{W_o}{W_1})$$

Since this moment the positive term of the acceleration vanishes and its value becomes

$$\frac{dV}{dt} = -g_o.$$

3.4 The computation of a trajectory consists of two parts: powered flight and coasting. Assume the initial conditions $y_o = 0$, $V_o = 0$.

In powered flight:

$$V = \frac{dy}{dt} = -g_o t + u\ \ln \frac{1}{1 - Kt}$$; assuming that the take off was at the ground level, i.e., $y = 0$, we obtain the altitude y at any time t after the ignition.

$$y = -\frac{g_o t^2}{2} + u \int_o^t \ln\ (\frac{1}{1 - Kt})\ dt,$$

$$\int_o^t \ln\ (1 - Kt)\ dt = t\ \ln\ (1 - Kt)\Big]_o^t - \int^t \frac{-Kt}{1 - Kt}\ dt =$$

$$= t\ \ln\ (1 - Kt) - t + \int_o^t \frac{1}{1 - Kt}\ dt =$$

$$= t\ \ln\ (1 - Kt) - t - \frac{1}{K}\ \ln\ (1 - Kt)\Big]_o^t =$$

$$= (t - \frac{1}{K})\ \ln\ (1 - Kt) - t = -t - \frac{(1 - Kt)}{K}\ \ln\ (1 - Kt);$$

hence

$$y = -\frac{g_o t^2}{2} + u \left[-(t \frac{1}{K}) \quad \ell n \; (1 - Kt) + t \right] \qquad (3\text{-}12)$$

Let the altitude at burn-out be y_o; the value of y_o can be found by setting

$$t = t_1 = \frac{1}{K} (1 - \frac{M_1}{M_o}) \quad ,$$

$$1 - Kt_1 = \frac{M_1}{M_o} \; ;$$

hence

$$y_o = -\frac{g_o t_1{}^2}{2} + u \left[t_1 + \frac{1 - Kt_1}{K} \ell n \; (1 - Kt_1) \right] =$$

$$= -\frac{g_o}{2} \left[\frac{1}{K} (1 - \frac{M_1}{M_o}) \right]^2 + \frac{u}{K} \left[(1 - \frac{M_1}{M_o}) - \frac{M_1}{M_o} \ell n \frac{M_o}{M_1} \right] (3\text{-}13)$$

Taking now the moment of burn-out inturn as t = 0, the velocity at any time t after the burn-out is given by the expression

$$V = V_{max} - g_o t$$

The maximum height is attained at the time t_2 after the burn-out

$$t_2 = \frac{V_{max}}{g_o} = -\frac{1}{K} (1 - \frac{M_1}{M_o}) + \frac{u}{g_o} \ell n \frac{M_o}{M_1} \; , \qquad (3\text{-}14)$$

The value of this height, beyond the burn-out point is

$$\Delta \; y_2 = \frac{V^2_{max}}{2g_o} \; .$$

Substitute the value of V_{max}.

$$\Delta \; y_2 = \frac{1}{2g_o} \left[u \; \ell n \frac{M_o}{M_1} - \frac{g}{K} (1 - \frac{M_1}{M_o}) \right]^2 \qquad (3\text{-}15)$$

the total maximum height will be: $y_o + \Delta \; y_2$.

Hence

$$y_{max} = -\frac{g_o}{2}\left[\frac{1}{K}\left(1-\frac{M_1}{M_o}\right)\right]^2 + \frac{u}{K}\left(1-\frac{M_1}{M_o}\right) -$$

$$- \frac{u}{K}\frac{M_1}{M_o}\ln\frac{M_o}{M_1}$$

$$+ \frac{u^2}{2g_o}\left(\ln\frac{M_o}{M_1}\right)^2 + \frac{g_o}{2}\left[\frac{1}{K}\left(1-\frac{M_1}{M_o}\right)\right]^2 -$$

$$- \frac{u}{K}\left(1-\frac{M_1}{M_o}\right)\ln\frac{M_o}{M_1}$$

$$= \frac{u^2}{2g_o}\left(\ln\frac{M_o}{M_1}\right)^2 + \frac{u}{K}\left(1-\frac{M_1}{M_o}\right) - \frac{u}{K}\ln\frac{M_o}{M_1}\ .$$

or,

$$y_{max} = \frac{I^2\ g_e^{\ 2}}{2\ g_o}\left(\ln\frac{W_o}{W_1}\right)^2 + \frac{I\ g_e}{K}\left[1-\frac{W_1}{W_o}-\ln\frac{W_o}{W_1}\right] \quad (3\text{-}16)$$

3.5 As an example the ascent equations have been applied to the computation of the take-off from the ground level for a lunar excursion module (LEM) of weight 4 tons. All weights in this problem are measured at the standard conditions on Earth.

Assume I = 300 sec, burning time $t_{b.o}$ = 40 sec, a constant propellant flow $\dot{W} = c = 20$ lb/sec;

$u = I\ g_e = (300)(32.174) = 9652$ ft/sec.

Hence $K = \frac{c}{W_o} = \frac{1}{400}\,\frac{1}{sec}$, $\frac{W_o}{W_1} = \frac{8000}{7200} = 1.111$, $\frac{W_1}{W_o} = \frac{7200}{8000} = 0.900$.

Burn-out conditions:

$$\dot{y}_{b.o} = V_{max} = -\frac{g_o}{K}\left(1 - \frac{W_1}{W_o}\right) + I\, g_e\, \ln \frac{W_o}{W_1} = -g_o\, t_{b.o.} +$$

$$+ I\, g_e\, (2.306)\, \log_{10} \frac{W_o}{W_1}$$

$$= -(40)(5.3) + (300)(32.174)(2.3026)\, \log_{10}(1.111)$$

$$= -212 + (9650)(2.3026)(0.04571) = -212 + 1017$$

$$= 805 \text{ ft/sec.}$$

Altitude $y_{b.o}$ attained at burn-out:

$$y_{b.o} = -\frac{g_o\, t^2}{2} + I\, g_e \left[t - \left(\frac{1}{K} - t\right) \ln \frac{W_o}{W_1} \right]$$

$$= -\frac{(5.3)(1000)}{2} + (9650)\left[40 - (400 - 40)\ln 1.111 \right]$$

$$= -4240 + (9650)\left[40 - (360)(.1052) \right] =$$

$$= -4240 + (9650)(40 - 37.9) = -4240 + 20260$$

$$= 16020 \text{ ft.}$$

Maximum acceleration attained at burn-out

$$\frac{\ddot{y}}{g_e} = -\frac{g_o}{g_e} + K\,\frac{W_o}{W_1}\, I =$$

$$= -0.164 + \frac{300}{400}\,\frac{W_o}{W_1} = -0.164 + (0.75)(1.111)$$

$$= 0.670$$

$$\ddot{y} = (0.670)(32.174) = 21.55 \text{ ft/sec}^2$$

The time of coasting to the maximum altitude

$$\Delta t = \frac{V_{max}}{g_o} = \frac{806}{5.3} = 152 \text{ sec.}$$

total time of ascent $t_{max} = t_{b.o} + \Delta t = 40 + 152 = 192$ sec.

extra height beyond the b.o.

$$\Delta y = \frac{V_{max}{}^2}{2g} = \frac{(805)^2}{10.6} = \frac{648025}{10.6} = 61100 \text{ ft.}$$

total $y_{max} = y_{b.o} + \Delta y = 16020 + 61100 = 77120$ ft.

This result is checked by using the equation (3-16) which now assumes the form

$$y_{max} = (8.78) \times 10^6 \left(\ln \frac{W_o}{W_1}\right)^2 +$$

$$+ (3.86) \times 10^6 \left(1 - \frac{W_1}{W_o} - \ln \frac{W_o}{W_1}\right).$$

$$y_{max} = \frac{(9650)^2}{10.6} (.1052)^2 + (9650)(400)(1 - 0.9 - 0.1052)$$

$$= \frac{(9.312) \times 10^7 \times 1.1067}{10.6 \times 10^2} -$$

$$- (9.65) \times (4) \times 10^6 \times (5.2) \times 10^{-3}$$

$$= 9.72 \times 10^4 - 2.06 \times 10^4 = 76600 \text{ ft.}$$

The same pattern is applied to other values of $\frac{W_o}{W_1}$ and the results are presented in the Table IV. All computations have been performed with slide rule accuracy, this explains the small discrepancy in the values of y_{max}.

TABLE IV

$I = 300$ sec., $\dot{W} = 20$ lb/sec., $K = \frac{1}{400}$

$\frac{W_o}{W_1}$	1.053	1.111	1.143	1.177	1.250	1.333	
$\ln \frac{W_o}{W_1}$	0.0516	0.1052	0.1340	0.1632	0.2234	0.2873	
$t_{b.o.}$	20	40	50	60	80	100	sec.
$\dot{y}_{b.o.}$	392	805	1028	1258	1750	2242	ft/sec.
$y_{b.o.}$	2800	16020	23310	33900	65040	106600	ft.
t_{max}	94	192	244	297	410	524	sec.
Δy	14500	61100	99700	149200	289200	474500	ft.
y_{max}	17300	77120	122010	183100	354240	581100	ft.
y_{max}	3.28	14.62	25.2	34.7	67.1	109	stat. miles
$\ddot{y}_{max}$	20.14	21.55	22.3	23.12	24.9	26.9	ft/sec^2
$\frac{\ddot{y}_{max}}{g_e}$	.626	.670	.694	.719	.774	.836	
y_{max}	17000	76600	123000	182200			check by (3-16) ft.

CHAPTER IV

Vertical Descent and Landing Constant Thrust

4.1 The problem is not a simple reversal of the take-off problem, but presents some different aspects.

Assume that at a certain altitude y_0 the horizontal velocity of the vehicle has been killed, but it still possesses a vertical velocity $\dot{y}_0$. This velocity plus the velocity acquired in the free fall from the altitude y_0 must be reduced to zero at the altitude just above the ground.

A particular case is when $\dot{y}_0 = 0$. It is shown in the next Chapter that in order to minimize the propellant comsumption it would be advisable to let the vehicle to drop freely for considerable time, and then to apply one or few bursts of thrust. This, however, implies a re-ignition of the engine and the safety considerations require the avoiding of such procedure, as the failure of the engine to ignite would be fatal, this consideration has precedence over the fuel economy. Hence we examine first the descent under continuous thrust, which may or may not be modulated.

Consider first the case of constant thrust and a constant propellant flow $\dot{W} = K\, W_0$. Initial conditions are y_0 and $\dot{y}_0 = V_0$. Final conditions are: $y_f = 0$, $\dot{y} = 0$. More properly y_f is a small altitude above the ground, at which the vehicle may hover, giving to the pilot the opportunity to select another landing place if necessary. At any time t since the beginning of the maneuver we have

$$\ddot{y} = - g_0 + \frac{I\ K}{1 - Kt} \qquad (1)$$

First and second integrations yield

$$\dot{y} = V_0 - g_0 t + I g_e \ln \frac{1}{1 - Kt} \quad \cdot \tag{2}$$

$$y = V_0 t - \frac{g_0 t^2}{2} + I g_e \left[t - (\frac{1}{K} - t) \ln \frac{1}{1 - Kt} \right] + + y_0 \tag{3}$$

The substitution of final conditions into the equations (2) and (3) yields two equations which theoretically can be solved to find K and t, that is the rate of fuel consumption and the time of descent.

Let $\ln \frac{1}{1 - Kt} = X$,

$$V_0 - g_0 t + I g_e X = 0 \quad , \tag{4}$$

$$V_0 t - \frac{g_0 t^2}{2} + I g_e \left[t - (\frac{1}{K} - t) X \right] + + y_0 = 0 \tag{5}$$

Note that $X > 0$

From (4) $$X = - \frac{V_0 - g_0 t}{I g_e} \tag{6}$$

Substitute into (5)

$$V_0 t - \frac{g_0 t^2}{2} + y_0 + I g_e t + (\frac{1}{K} - t)(V_0 - g_0 t) = 0 \tag{7}$$

Solving (6) and (7) with respect to K

$$\frac{1}{K} = t + \frac{\frac{g_0 t^2}{2} - (I g_e + V_0)t - y_0}{V_0 - g_0 t} =$$

$$\frac{1}{K} = \frac{\frac{g_o t^2}{2} + I g_e t + y_o}{g_o t - V_o}$$

$$K = \frac{g_o t - V_o}{\frac{g_o t^2}{2} + I g_e t + y_o} = f(t) \tag{8}$$

and from (6)

$$\frac{1}{1 - Kt} = e^{\frac{g_o t - V_o}{I g_e}}$$

$$K = \frac{1}{t}\left[1 - e^{-\frac{g_o t - V_o}{I g_e}} \right] = \phi(t). \tag{9}$$

Plotting curves $K = f(t)$ and $K = \phi(t)$ their intersections yields the values for t and K. The method is illustrated by considering a vehicle descending from the altitudes y_o = 5,000 ft, 10,000 ft, 20,000 and 40,000 ft, to the lunar surface. In all cases it has been assumed V_o = - 400 ft/sec and I = 300 sec. The curves for the solution of equation (7) have been computed in Tables V, VI, VII and VIII, and the results presented on the Graph A, where the solutions have been obtained.

TABLE V

y_o = 10,000 ft. V_o = - 400 ft/sec. $I\ g_e$ = 9652 ft/sec.

1.	t	100	150	200	250	sec.
2.	$g_o\ t$	530	795	1060	1324	ft/sec.
3.	$g_o\ t - V_o$	0.930	1.195	1.400	1.724	$\times 10^3$ ft/sec.
4.	$\frac{g_o\ t^2}{2}$	2.65	5.96	10.6	16.51	$\times 10^4$ ft.
5.	$I\ g_e\ t + y_o$	97.52	145.85	194.04	242.4	$\times 10^4$ ft.
6.	(4) + (5)	100.17	151.81	204.64	258.91	$\times 10^4$ ft.
7.	$K = \frac{(3)}{(6)}$	9.27	7.87	7.13	6.94	$\times 10^{-4}\ \frac{1}{\text{sec.}}$
8.	$\frac{(3)}{I\ g_e}$	0.0964	.1239	.1512	.1787	
9.	$e^{-(8)}$	0.907	.883	.860	.837	
10.	1 - (9)	0.093	.117	.140	.163	
11.	$K = \frac{(10)}{(1)}$	9.3	7.80	7.0	6.52	$\times 10^{-4}\ \frac{1}{\text{sec}}$.
	From the graph $K = 8.93 \times 10^{-4}\ \frac{1}{\text{sec}}$, t = 111 sec. Kt = 0.0992					

TABLE VI

y_o = 5000 ft. V_o = - 400 ft/sec. $I\ g_e$ = 9652 ft/sec.

1.	t	50	100	150	sec.
2.	$g_o\ t$	265	530	795	ft/sec.
3.	$g_o\ t - V_o$	0.665	0.930	1.195	x 10^4 ft/sec.
4.	$\frac{g_o\ t^2}{2}$	0.663	2.65	5.96	x 10^4 ft.
5.	$I\ g_e\ t + y_o$	48.76	97.02	145.28	x 10^4 ft.
6.	(5) + (4)	49.42	99.67	151.24	x 10^4 ft.
7.	$\frac{(3)}{(6)} = K$	13.47	9.32	7.90	x 10^{-4} $\frac{1}{\text{sec.}}$
8.	$\frac{(3)}{I\ g_e}$	0.0689	.0964	.1239	
9.	$e^{-(8)}$	.932	.907	.883	
10.	1 - (9)	.068	.093	.117	
11.	$\frac{(10)}{(1)} = K$	13.6	9.3	7.80	x 10^{-4} $\frac{1}{\text{sec}}$

t = 98 sec. $K = 9.4 \times 10^{-4}\ \frac{1}{\text{sec.}}$ Kt = 0.0921

TABLE VII

$y_o = 20{,}000$ ft. $V_o = -400$ ft/sec. $I\ g_e = 9652$ ft/sec.

1.	t	100	150	200	250	sec.
2.	$g_o\ t$	530	795	1060	1324	ft/sec.
3.	$g_o\ t - V_o$	0.930	1.195	1.460	1.724	$\times 10^3$ ft/sec.
4.	$\frac{g_o\ t^2}{2}$	2.65	5.96	10.6	16.51	$\times 10^4$ ft.
5.	$I\ g_e\ t + y_o$	98.52	146.85	195.04	243.4	$\times 10^4$ ft.
6.	(4) + (5)	101.17	152.81	205.64	259.91	$\times 10^4$ ft.
7.	$\overset{\circ}{K} = \frac{(3)}{(6)}$	9.19	7.84	7.10	6.64	$\times 10^{-4}$ ft.
8.	$\frac{(3)}{I\ g_e}$	0.0964	.1239	.1512	.1787	
9.	$e^{-(8)}$	0.907	.883	.860	.837	
10.	1 - (9)	0.093	.117	.140	.163	
11.	$K = \frac{(10)}{(1)}$	9.3	7.80	7.0	6.52	$\times 10^{-4}$ ft.

$t = 134$ sec. $K = 8.2 \times 10^{-4}\ \frac{1}{\text{sec}}$ $Kt = 0.109$

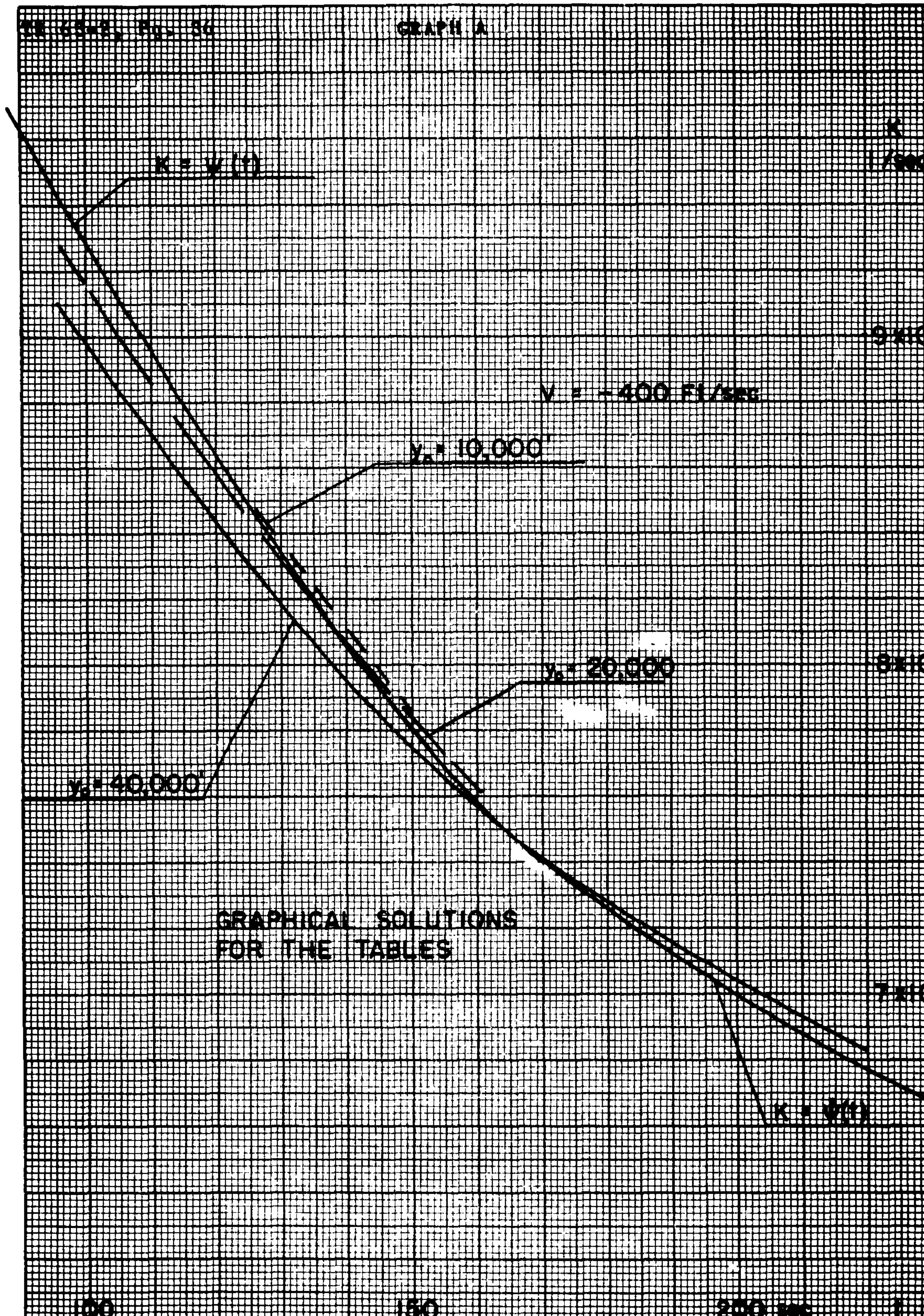
TABLE VIII

$y_o = 40000$ ft. $V_o = 400$ ft/sec.

1	t	100	150	200	250	sec.
(3)	$g_o t - V_o$	0.930	1.195	1.460	1.724	$\times 10^3$ ft/sec.
(6)	(5) + (4)	103.17	154.81	207.64	261.91	$\times 10^4$ ft.
(7)	$K = \frac{(3)}{(6)}$	9.00	7.74	7.04	6.59	$\times 10^{-4}\ \frac{1}{sec}$
(11)	$K = \frac{(10)}{(1)}$	9.30	7.80	7.0	6.52	$\times 10^{-4}$ sec.

$t = 168$ sec. $k = 7.44 \times 10^{-4}\ \frac{1}{sec}$

$Kt = 0.125$


GRAPH A
K
1/sec
K = ψ (t)
9×10
V = −400 Ft/sec
y0 = 10,000'
8×10
y0 = 20,000
y0 = 40,000'
GRAPHICAL SOLUTIONS
FOR THE TABLES
7×10
K = ψ(t)
100
150
200 sec
t

The Tables V, VI, VII and VIII provide data for graphical solutions of equations (4) and (5). For all altitudes one common curve $K = \psi(t)$ is plotted, which does not depend on y_0. Then for any value of y_0 a curve $K = f(t)$ shown by equation (8) is plotted. The intersection of $\psi(t)$ and $f(t)$ yields the value of K and that of t_1 - time of descent. Some obtained results have been checked by substitution into the equations.

(1)	t	134	111	98	sec.
(2)	K	8.2	8.93	9.4	$\times 10^{-4} \frac{1}{\text{sec}}$
(3)	Kt	0.109	0.0992	0.0921	
(4)	$1 - Kt$	0.891	0.9008	0.9079	
(5)	$\frac{1}{1 - Kt}$	1.1223	1.1101	1.1014	
(6)	$\log_{10} \frac{1}{1 - Kt}$	0.04999	.04532	.04198	
(7)	(2.3026)(6)	0.1151	0.1043	0.0967	
(8)	$g_o\ t$	710	588	520	ft/sec
(9)	$g_o\ t - V_o$	1110	988	920	ft/sec
(10)	$\frac{(9)}{I\ g_e}$	0.1150	0.1024	0.954	

This table served to check the graphical solutions by substitution into the equation (6) written as

$$\ell n \frac{1}{1 - Kt} = \frac{g_o\ t - V_o}{I\ g_e}$$

Line (~~6~~ 7) yields the values for the left side and the line (10) for the right side of this equation.

Check of the equation (5), written as:

$$\frac{g_o\ t^2}{2} - y_o - V_o t = I\ g_e \left[t - \frac{1 - Kt}{K} \ell n \left(\frac{1}{1 - Kt} \right) \right]$$

we obtain for the left side in the case y_o = 10000 ft.

$$\frac{(5.3)\ (12321)}{2} - 10{,}000 + 400 \times 111 =$$

$$= 32630 - 10{,}000 + 44400 = 67030 \text{ ft;}$$

and for the right side:

$$I\ g_e \left[t - \frac{1 - Kt}{K} \ell n \left(\frac{1}{1 - Kt} \right) \right] =$$

$$= 9652 \left[111 - \frac{0.891 \times 10^4}{8.93} (0.1043) \right] =$$

$$= 9652 \left[111 - \frac{8.91}{8.93} \times 10^3\ (0.1043) \right] =$$

$$= 9652\ (111 - 104.07) =$$

$$= (9652)\ (6.93) = 68200 \text{ ft.}$$

The agreement is fairly good for the slide rule accuracy. The results of the computations are summarized in the following table and graph, where t is the time of descent from the initial altitude y_o to y = 0, $Kt = \frac{W_o - W_1}{W_o}$ is the ratio of the propellant consumed during the maneuver to the initial weight of the vehicle at $y = y_o$. The magnitude of the initial and final accelerations are computed from the equation

$$\frac{\ddot{y}}{g_e} = -\ 0.164 + KI\ \frac{W_o}{W_1}\ ,$$

where

$$W_1 = W_o\ (1 - Kt)\ ,$$

for t = 0, $\frac{\ddot{y}}{g_e} = -\ 0.164 + KI$ = initial acceleration in terms of the Earth standard gravity g_e.

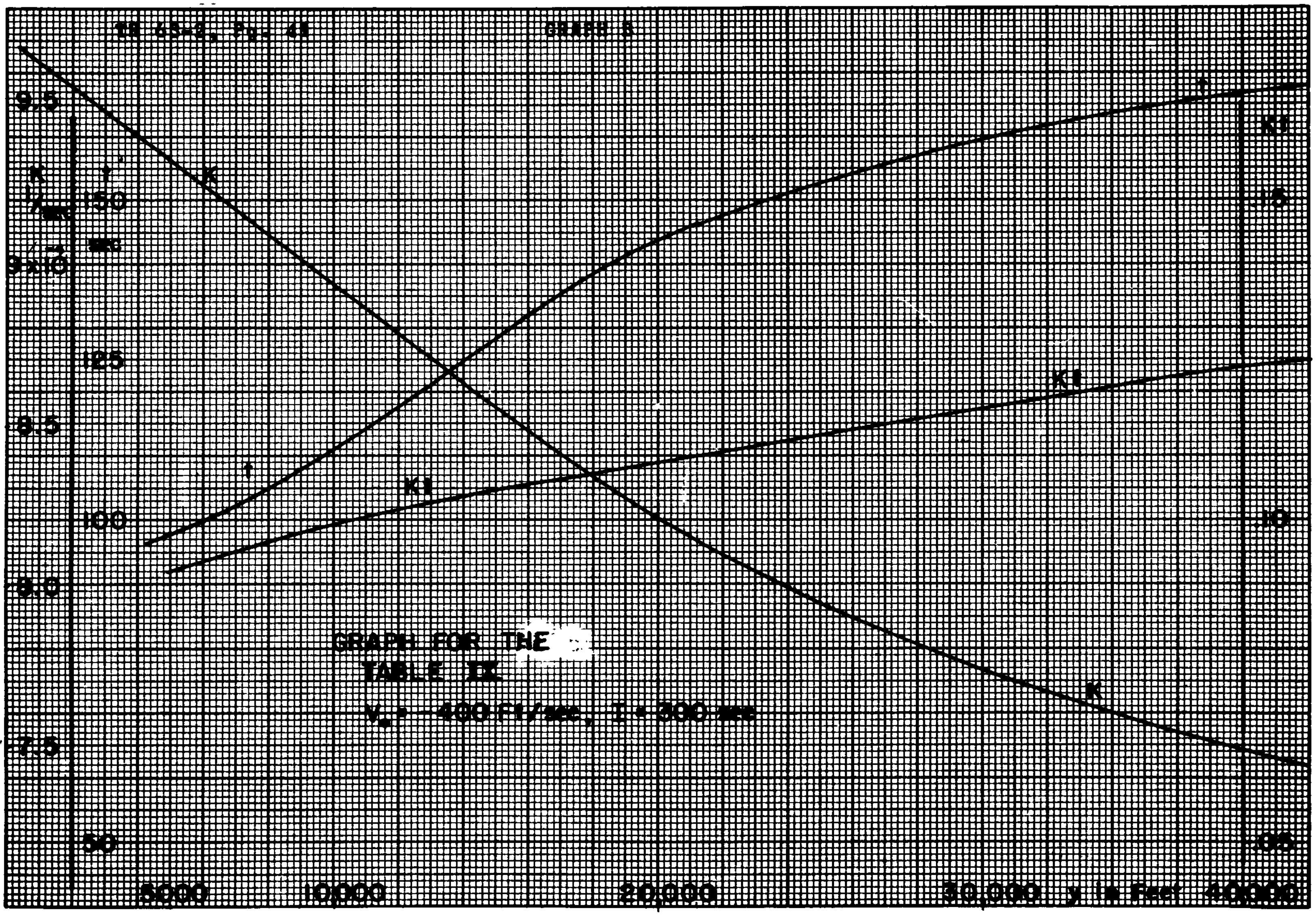
For each landing case the rate of fuel consumption and the thrust are constant. This implies that whatever is the fuel flow, the specific thrust I remains constant. The ratio of thrust to the initial weight is $\frac{F_o}{W_o} = \frac{\dot{W}}{W_o}\ I = KI$. The weight here is the standard Earth weight of the vehicle.

The results obtained for descent from different altitudes are summarized in the Table IX and presented graphically on the Graph B.

TABLE IX

$V_o = -\ 400$ ft/sec. $I = 300$ sec. for different altitudes

1.	y_o	5000	10000	20000	40000	ft.
2.	t	98	111	134	168	sec.
3.	K	9.4	8.93	8.2	7.44	$\frac{1}{sec} \times 10^{-4}$
4.	Kt	.0921	.0992	.109	.125	
5.	$1 - Kt$	.9079	.9008	.891	.875	
6.	$\frac{w_o}{w_1}$	1.1014	1.1101	1.1223	1.1429	
7.	$KI = F_o/w_o$	.282	.268	.246	.223	
8.	$(\ddot{y}_o/g_e)$	.118	.104	.082	.059	
9.	$KI \frac{w_o}{w_1}$	.311	.298	.276	.256	
10.	$\ddot{y}_1/g_e$	.145	.134	.112	.092	
11.	$\frac{F_o}{w_o} \frac{g_e}{g_o}$	1.720	1.632	1.500	1.360	



GRAPH FOR THE
TABLE IX
9.5
150
125
8.5
100
9.0
7.5
50
K
K1
.10
.05
5000
10,000
20,000
30,000
y in Feet
40000

Note that the thrust F_0 is compared to the initial weight of the vehicle at the Earth's standard gravity. If the lunar initial weight is considered, then

$$\frac{F_0}{(w_0)_e} = \frac{F_0}{(w_0)_E} \frac{g_e}{g_0} = (\frac{F_0}{w_0}) \frac{1}{0.164}$$

We have investigated the case of a descent from different altitudes, with the same initial vertical velocity. Consider now the case of a vertical descent from the same altitude but with different initial velocities V_0. For an example the altitude y_0 = 10,000 ft has been selected, and for V_0 the downward values 0, -400, -800, -1000 ft/sec assumed. The equations (8) and (9) have been used for plotting the curves $K = f(t)$ and $K = \phi(t)$; the coordinates for these curves are computed in the Tables X, XI, XII and V, the curves are plotted on Graphs C and D, yielding graphical solutions. The results are summarized on the Table XIII; then K, Kt, and final acceleration $\frac{\ddot{y}}{g_e}$ are plotted versus the initial velocity V_0 on the Graph E, which shows that for each initial altitude there is an initial V_0 yielding a minimum propellant consumption

$Kt = \frac{\text{propellant weight}}{\text{initial weight}}$. In the considered example this minimum occurs for y_0 = 10,000 ft, V_0 = - 570 ft/sec.

TABLE X

$y_o = 10{,}000$ ft. $V_o = 0$

1.	t	250	300	360	400	sec.
2.	$g_o\ t$	1325	1590	1909	2120	ft/sec
3.	$g_o\ t - V_o$	1.325	1.59	1.909	2.120	x 10^3 ft/sec
4.	$\frac{g_o\ t^2}{2}$	16.55	23.85	34.34	42.4	x 10^4 ft.
5.	$1\ g_e\ t + y_o$	242.4	290.56	348.5	387.08	x 10^4 ft.
6.	(4) + (5)	258.95	314.41	382.34	429.48	x 10^4 ft.
7.	$K = \frac{(3)}{(6)}$	5.115	5.054	4.99	4.94	x 10^{-4} $\frac{1}{sec}$
8.	$\frac{(3)}{I\ g_e}$	.1372	0.1649	.1978	.2196	
9.	$e^{-(8)}$	.8718	.8480	.8205	.8028	
10.	1 - (9)	.1282	.1520	.1795	.1972	
11.	$\frac{(10)}{(1)}$	5.128	5.067	4.99	4.93	x 10^{-4} $\frac{1}{sec}$

$t = 360$ sec. $K = 4.99 \times 10^{-4}\ \frac{1}{sec}$ $Kt = 0.1798$

TABLE XI

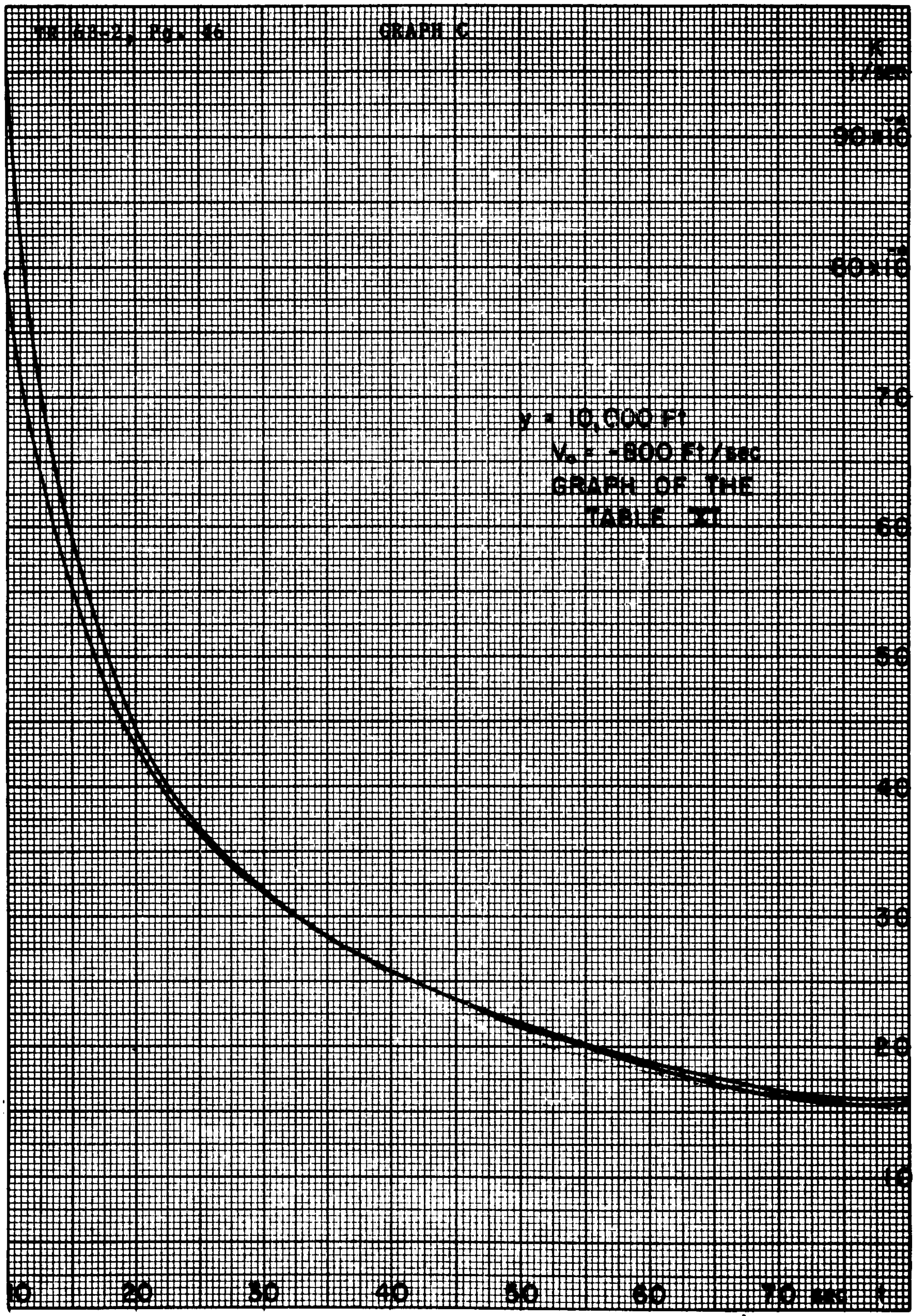
y_o = 10,000 ft. V_o = - 800 ft/sec.

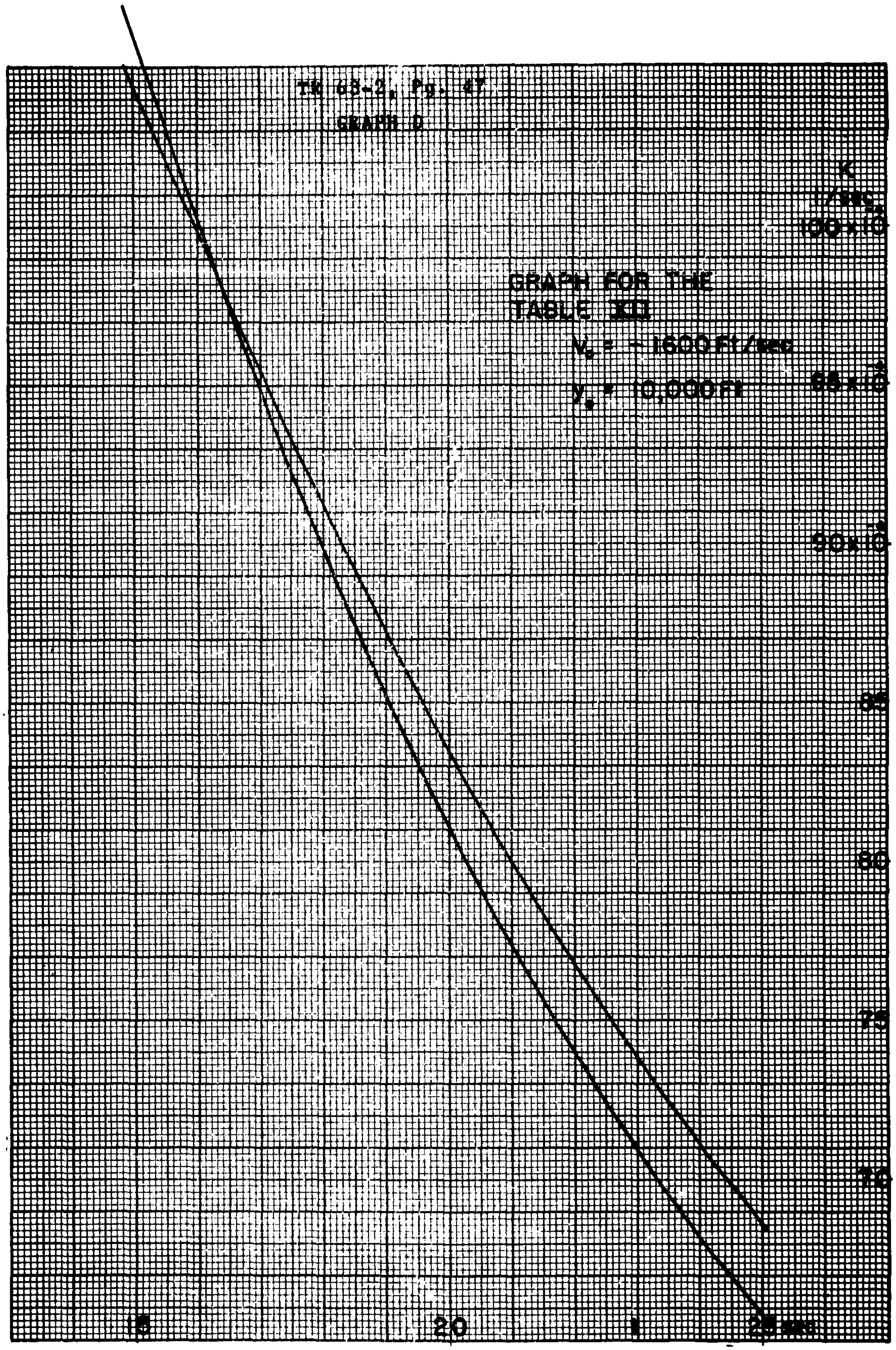
1.	t	10	25	50	75	sec.
2.	$g_o t$	53	132.5	265	397.5	sec.
3.	$g_o t - V_o$	.853	.9325	1.065	1.1975	x 10^3 ft/sec
4.	$\frac{g_o t^2}{2}$	.0265	.1658	.6625	1.49	x 10^4 ft.
5.	$I g_e t + y_o$	10.65	25.13	49.26	73.40	x 10^4 ft.
6.	(4) + (5)	10.68	25.30	49.92	74.89	x 10^4 ft.
7.	$K = \frac{(3)}{(6)}$	79.9	36.85	21.34	16.00	x 10^{-4} $\frac{1}{sec}$
8.	$\frac{(3)}{I g_e}$	.0883	.0967	.1103	.1241	
9.	$e^{-(8)}$	.9155	.9078	.8956	.8833	
10.	1 - (9)	.0945	.0922	.1044	.1167	
11.	$K = \frac{(10)}{(1)}$	94.50	36.88	20.88	15.54	x 10^{-4} $\frac{1}{sec}$

TABLE XII

$y_o = 10{,}000$ ft. $V_o = -1600$ ft/sec.

1.	t	10	15	20	25	50	75	100	sec.
2.	$g_o\ t$	53	79.5	106	132.5	265	397.5	530	ft/sec
3.	$g_o\ t - V_o$	1.653	1.6795	1.706	1.7325	1.865	1.9975	2.130	x 10^3 ft/sec
4.	$\frac{g_o\ t^2}{2}$	.0265	.596	.1060	.1658	.6625	1.49	2.65	x 10^4 ft.
5.	$I\ g_e\ t + y_o$	10.65	15.48	20.304	25.13	49.267	73.40	97.52	x 10^4 ft.
6.	(4) + (5)	10.68	16.08	20.41	25.30	49.93	74.89	100.17	x 10^4 ft.
7.	$K = \frac{(3)}{(6)}$	155.0	104.4	83.52	68.5	37.00	26.72	21.24	$10^{-4}\ \frac{1}{sec}$
8.	$\frac{(3)}{I\ g_e}$	.1713	.1740	.1768	.1797	.1931	.2070	.2206	
9.	$e^{-(8)}$	.8426	.8403	.8379	.8355	.8244	.8129	.8020	
10.	1 - (9)	.1574	.1597	.1621	.1645	.1756	.1871	.1980	
11.	$K = \frac{(10)}{(1)}$	157.4	106.4	81.05	65.80	35.12	24.96	19.80	x $10^{-4}\ \frac{1}{sec}$


GRAPH C
K
y = 10,000 Ft
V_o = -800 Ft/sec
GRAPH OF THE
TABLE XI
70
60
50
40
30
20
10
20
30
40
50
60
70 sec



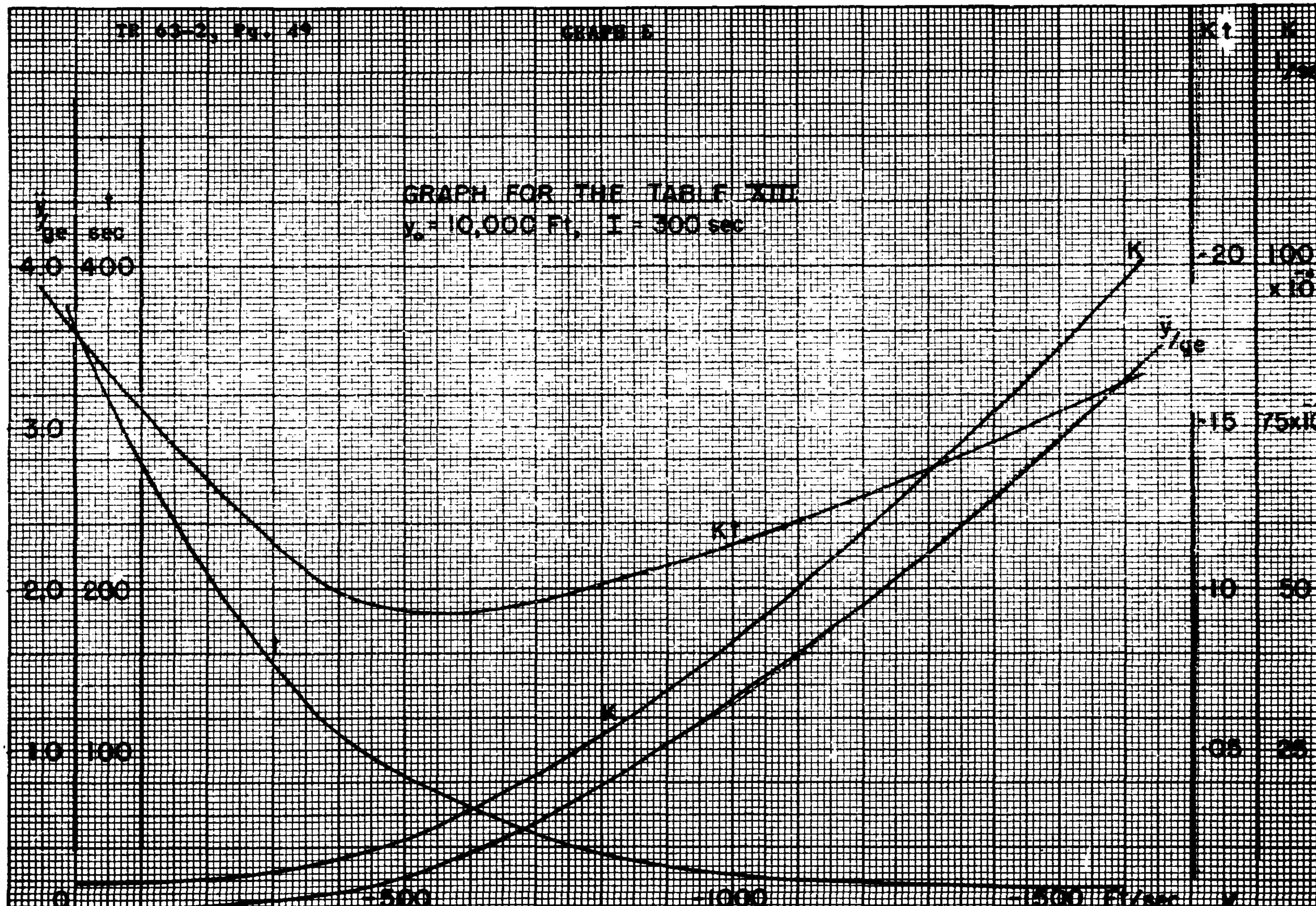
TR 63-2, Pg. 47
GRAPH D
K
1/sec
100 x 10⁻³
GRAPH FOR THE
TABLE XII
V_o = − 1600 Ft/sec
y_o = 10,000 Ft
95 x 10⁻³
90 x 10⁻³
85
80
75
70
15
20
25 sec

TABLE XIII

y_o = 10,000 ft.   I = 300 sec.

1.	V_o	0	-400	-800	-1600	ft/sec.
2.	t_1	360	111	36.5	16.4	sec.
3.	K	4.99	8.93	27.8	98.0	$\times 10^{-4} \frac{1}{sec}$
4.	$K t_1$	.1798	0.0992	0.1015	0.1608	
5.	$1 - K t_1$	.8202	.9008	.8985	.8392	
6.	$\frac{W_o}{W_1}$	1.2192	1.1101	1.1130	1.1916	
7.	$KI = \frac{F_o}{W_o}$	.1497	.268	.834	2.94	
8.	$\frac{F_o}{W_o} \frac{g_e}{g_o}$	.913	1.632	5.08	17.91	
9.	$\frac{\ddot{y}_1}{g_e}$	0.0185	.134	.779	3.34	

GRAPH 5


GRAPH FOR THE TABLE XIII
y_o = 10,000 Ft, I = 300 sec
$\ddot{y}/g_e$
t sec
4.0 400
3.0
2.0 200
1.0 100
0
-500
-1000
-1500 Ft/sec
V
K
Kt
t
$\ddot{y}/g_e$
Kt
.20
.15
.10
.05
K 1/sec
100 x 10^{-4}
75x10^{-4}
50
25

CHAPTER V

Intermittent and Modulated Thrust

5.1 Assume that the horizontal velocity of the vehicle has been killed at the altitude y_0. Its vertical velocity is $\dot{y} = V_0$, which may be zero in particular cases. In the previous chapter a vertical descent under a constant continuous thrust has been considered, without the stalling and restarting the engine. Assume now that the engine has been switched off, the vehicle in free fall will reach the altitude

$$y_1 = y_0 + V_0\ t - \frac{g_0\ t^2}{2} \quad ,$$

with a vertical velocity $\dot{y}_1 = V_0 - g_0\ t$. At this altitude the engine will be again ignited, then the rate of the propellant flow will be computed as it has been shown in the Chapter IV in order to reach the ground with zero velocity.

Without embarking on a generalized optimization problem let us compare two methods of descent from the point of view of propellant economy. Consider a descent from the altitudes y_0 = 20,000 ft and y_0 = 40,000 ft, assuming the initial vertical V_0 = - 400 ft/sec for both cases. Assume that the ignition occurs when the vehicle reaches the altitude y_1 = 10,000 ft. Graph on Figure E enables us to find K and fuel consumption

$$\frac{W_0 - W_1}{W_0} = Kt \text{ required to reach the ground.}$$

Powered descent from 20,000 ft: (as given in the Table IX)

Propellant consumption $Kt = 0.109$ of the initial weight

Time of descent $t = 134$ sec.

Maximum acceleration $= .112\ g_e$

Drop from $y_o = 20{,}000$ ft to $y_1 = 10{,}000$ ft., time of the fall is computed from the equation

$$\Delta y = -\frac{g_o\ t^2}{2} + V_o\ t,$$

where

$$\Delta y = y_1 - y_o = -\ 10{,}000 \text{ ft.}$$

Hence

$$g_o\ t^2 - 2\ V_o\ t + 2\ \Delta y = 0;$$

$$t = \Delta t = \frac{V_o \pm \sqrt{V_o^2 - 2g\ \Delta y}}{g_o}$$

Considering only the value $\Delta t > 0$, we obtain

$$\Delta t = \frac{-\ 400 + \sqrt{400^2 + 2 \times 5.3 \times 10{,}000}}{5.3} = \frac{116.3}{5.3} =$$

$$= 21.95 \text{ sec.}$$

The velocity attained is

$$V_1 = -\sqrt{V_o^2 - 2g_o\ \Delta y} = -\ 516.3 \text{ ft/sec.}$$

In the subsequent powered descent we obtain from the Graph E based on Table XIII.

$$t = 81 \text{ sec.} \quad , \quad t + \Delta t \cong 103 \text{ sec.}$$

$$Kt = 0.0935 \qquad K = 12.1 \times 10^{-4} \frac{1}{\text{sec}}.$$

$$\text{Max. accel.} = 0.24\ g_e$$

Both methods are compared in the following table:

Table XIV

Descent from $y_o = 20{,}000$ ft., $V_o = -400$ ft/sec.
I = 300 sec.

	Constant Thrust	Intermittent Thrust	
t Total time of descent	134	103	sec
$\frac{\Delta W}{W_o} = Kt$	0.109	0.0935	
$\ddot{y}/g$	.112	.24	
K	8.2	12.1	$\times 10^{-4} \frac{1}{\text{sec}}$

Hence the second method shows a not negligible fuel economy, with an acceleration well below the Earth's gravity.

$$\text{fuel economy} = \frac{109 - 93.5}{109} = \frac{16.5}{109} = 15.1 \text{ p.c.}$$

The similar computation applied to a powered descent from the altitude 40,000 ft. yields:

$$\Delta t = 56.8 \text{ sec.} \qquad V_1 = 691.4 \text{ ft/sec.}$$

Table XV

Descent from y_0 = 40,000 ft., V_0 = - 400 ft/sec.
I = 300 sec.

	Constant Thrust	Intermittent Thrust	
t	168	56.8 + 50 = 106.8	sec.
$\frac{\Delta W}{W_0} = Kt$	.125	.096	
$\ddot{y}/g_e$	.092	0.54	
K	7.44	20.8	$\times\ 10^{-4}\ \frac{1}{\text{sec}}$

In this case the economy of propellant is even better, being

$$\frac{.125\ -\ .096}{.125} = \frac{.29}{125} = 23.2 \text{ p.c.}$$

The max. acceleration is still only about half of the Earth's gravity.

5.2 Modulated Thrust

In order to avoid the stalling of the engine and to obtain some fuel economy in conditions similar to those with the intermittent thrust, a modulated thrust can be used. Modulation

is obtained by varying the flow of propellant. This involves extra difficulties, as this flow should be continuously modulated. Besides we assume a constant specific thrust which is the measure of conversion of the thermal energy released in a chemical reaction into the Kinetic energy of the mass flow. This conversion is optimum for a certain value of the mass flow corresponding to a given size and shape of the nozzle and combustion chamber. The value of I may drop if these conditions are not fulfilled, hence an adjustable nozzle will be required. This entails extra complexity of design, which should be rather avoided.

So we have either to assume that I = const., or provide the diagram of its variation with the mass flow. In this section we assume I = const = 300 sec., irrespectively of the magnitude of $\dot{W}$ or of the size of the nozzle.

The variation of the thrust F is assumed to follow the law $\frac{F}{W} = \frac{F_0}{W_0}$ = const; where W is the weight of the vehicle at a given time t. Since the gravity acceleration is assumed to be constant, the decrease of the weight is due only to the propellant flow. W_0 is the weight of the vehicle at the beginning of the maneuver, that is at $y = y_0$. Its value is the initial weight of the vehicle measured on the lunar surface, (not on the Earth). For a given altitude y_0 and vertical velocity V_0 we have to find the time of descent t_1, the modulation of $K = \frac{\dot{W}}{W_0}$ as a function of time and the ratio of the propellant consumed to the initial

weight of the vehicle: $\frac{\Delta W}{W_0} = \int_0^{t_1} K dt$. We start with the equation of motion

$$\frac{W}{g_0}\ddot{y} = F - W$$

or

$$\ddot{y} = \left(\frac{F}{W} - 1\right) g_0 = \left(\frac{F_0}{W_0} - 1\right) g_0 = \text{const.} \tag{5-1}$$

The first and second integrations yield

$$\dot{y} = g_0\left(\frac{F_0}{W_0} - 1\right) t + \dot{y}_0 = \left(\frac{F_0}{W_0} - 1\right) g_0\, t + V_0 \tag{5-2}$$

$$y = \left(\frac{F}{W} - 1\right)\frac{g_0\, t^2}{2} + V_0\, t + y_0 \tag{5-3}$$

At the ground level we ought to have

$y = 0$, $\dot{y} = 0$ and obtain two equations with the unknowns

t_1 and $\frac{F_0}{W_0}$.

From (5-2)

$$t_1 = -\frac{V_0}{\left(\frac{F_0}{W_0} - 1\right) g_0}$$

The (5-3) yields

$$\frac{V_0^{\,2}}{2\left(\frac{F_0}{W_0} - 1\right) g_0} - \frac{V_0^{\,2}}{\left(\frac{F_0}{W_0} - 1\right) g_0} + y_0 = 0$$

hence

$$\frac{F_0}{W_0} = \frac{V_0^{\,2}}{2\, y_0\, g_0} + 1 . \qquad \frac{F_0}{W_0} - 1 = \frac{V_0^{\,2}}{2\, y_0\, g_0}$$

and

$$t_1 = \frac{2y_0}{V_0} .$$

The rate $K = \frac{\dot{W}}{W_o}$ of the fuel flow can be found from the equation

$$W = W_o - \int_0^t \frac{\dot{W}}{W_o} dt \text{ , K is variable in this case}$$

$$F = I\dot{W} \frac{g_e}{g_o} \quad ; \quad \text{but} \quad \dot{W} = - \frac{dW}{dt}$$

so

$$F = - I \frac{g_e}{g_o} \frac{dW}{dt} \quad ; \quad \text{divide both sides by W.}$$

$$\frac{F}{W} = \frac{F_o}{W_o} = - I \frac{g_e}{g_o} \cdot \frac{dW}{dt} \cdot \frac{1}{W}$$

$$\frac{dW}{W} = - \frac{g_o}{g_e} \frac{1}{I} \cdot \frac{F_o}{W_o} dt$$

for any given time t which elapsed since the beginning of the maneuver we have

$$\ell n \frac{W_o}{W} = \frac{g_o}{g_e} \frac{1}{I} \frac{F_o}{W_o} t$$

and

$$W = W_o \, e^{- \frac{g_o}{g_e} \frac{1}{I} \frac{F_o}{W_o} t}$$

The amount of the propellant consumed since t = 0 till t = t is

$$\frac{W_F}{W_o} = \frac{W_o - W}{W_o} = 1 - e^{- \frac{g_o}{g_e} \frac{F_o}{W_o} \frac{t}{I}}$$

and

$$K = - \frac{1}{W_o} \frac{dW}{dt} = + \frac{g_o}{g_e} \frac{F_o}{W_o} \frac{1}{I} e^{- \frac{g_o}{g_e} \frac{F_o}{W_o} \frac{t}{I}}$$

As an example we compute the descent from $y_o = 40{,}000$ ft, $V_o = -400$ ft/sec, for $I = 300$ sec and compare the results with those for constant and intermittent thrust.

$$t_1 = -\frac{2\ y_o}{V_o} = \frac{2 \times 4 \times 10^4}{4 \times 10^2} = 200 \text{ sec.}$$

$$\frac{F_o}{W_o} = \frac{{V_o}^2}{2\ y_o\ g_o} + 1 = 1 + \frac{4 \times 4 \times 10^4}{2 \times 4 \times 10^4 \times 5.3} =$$

$$= \frac{4}{10.6} + 1 = 1.377$$

The final value of the variable thrust

$$\frac{F_1}{W_o} = \frac{F_1}{W_1}\frac{W_1}{W_o} = \frac{F_o}{W_o} \cdot \frac{W_1}{W_o} = \frac{F_o}{W_o}\, e^{-\frac{g_o}{g_e}\frac{F_o}{W_o}\frac{t_1}{I}} .$$

$$\frac{g_o}{g_e}\frac{F_o}{W_o}\frac{1}{I} = (0.164)(1.377)\left(\frac{1}{300}\right) = 7.53 \times 10^{-4}\ \frac{1}{\text{sec}}.$$

for $t_1 = 200$ sec, we have

$$e^{-7.53 \times 10^{-4} \times 2 \times 10^2} = e^{-1.506 \times 10^{-1}} =$$

$$= e^{-0.1506} = 0.8602$$

Hence

$$\frac{F_1}{W_o} = (1.377)(.8602) = 1.184$$

Propellant consumed in the descent

$$\frac{W_F}{W_o} = 1 - e^{-0.1506} = 1 - 0.8602 = .1398; \quad \frac{W_o}{W_1} = e^{-0.1506} = 1.1625$$

Acceleration is constant in this case: $\frac{\ddot{y}}{g_e} = - 0.164 + \frac{F_o}{W_o} \frac{g_o}{g_e} I =$

$= - 0.164 + 0.226 = 0.062.$

TABLE XVI

$K = 7.53 \times 10^{-4} \times e^{-7.53 \times 10^{-4} t}$

t	0	50	100	150	200	sec
$7.53\ t \times 10^{-4}$	0	0.03765	0.0753	0.1130	0.1506	
$e^{-7.53 \times 10^{-4} t}$	1	0.9630	.9275	.8932	.8602	
$K = \frac{\dot{W}}{W_o}$	7.53	7.25	6.99	6.73	6.48	$\times 10^{-4} \frac{1}{sec}$

The table provides the programming of the fuel flow during the descent from y = 40,000 ft. It is valid only if the specific thrust I = 300 sec. remains constant in spite of the varying conditions in the nozzle and combustion chamber. If, however, I is not constant, its variable values provided by the engine data must be introduced in the expressions for thrust and rate of propellant flow.

The results of the present investigation can be summarized in the following table.

TABLE XVII

Descent and Landing from $y = 40{,}000$ ft. with $V_o = -400$ ft/sec. and $I = 300 \frac{1}{sec} = Const.$

CASE	A	B	C
Method of descent	Continuous constant thrust	Free fall to $y = 10{,}000$ ft. followed by continuous constant thrust	Modulated thrust $\frac{F}{W} = \frac{F_o}{W_o}$
Total time of descent	168	106.8	200 sec
Propellant flow $K = \frac{\dot{W}}{W_o}$	7.44×10^{-4}	20.8×10^{-4}	Initial $= 7.53 \times 10^{-4} \frac{1}{sec}$ Final $= 6.48 \times 10^{-4} \frac{1}{sec}$
Total propellant consumption $\frac{W_F}{W_o}$	0.125	0.096	0.1398
Max. acceleration in terms of g_e	0.092	0.54	0.062 = constant

Hence a modulated thrust did not produce an expected reduction in propellant consumption, because the lower rate of fuel flow was offset by the length of the descent time. Besides, this case would require a continually adjustable nozzle in order to maintain a constant specific thrust. This would entail extra complexity of the design and a decrease in reliability. So, at least for the given initial conditions, the choice would be rather between the cases A and B.